Analytical Investigation of Near-Parabolic Lunar Trajectories between Moon and Cislunar Libration Point

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An approximate analytical theory is developed for analysis for near-parabolic trajectories between the moon and the cislunar-libration point (L_1). The analysis of the Earth as a perturbative influence on the moon-referenced trajectories involves the local regularization of the unperturbed and perturbed two-body problem and development of general Encke-type perturbation theory in the regularized domain. The linearized perturbation equations are demonstrated to be analytically integrable in the regularized domain. The analysis investigates velocity requirements at L_1 and the moon for passage in either direction and passing either in front of or behind the moon. The approximate analytic theory gives close agreement to representative numerical results from a digital computer.

Nomenclature

a = semimajor axis
E,F = eccentric, hyperbolic anomaly
F_v,F_s = system matrices, Eq. (23a), (27a)
$\mathbf{f}$ = force vector
$\mathbf{g}_v,\mathbf{g}_s$ = forcing vector, Eq. (23a), (27a)
$\mathbf{h}$ = angular momentum vector
$\mathbf{i},\mathbf{j},\mathbf{k}$ = unit vectors
L_1 = cislunar-libration point
$\mathbf{n},n$ = angular rotation vector, rate of Earth-moon system
p = semilatus rectum
$\mathbf{r},r$ = radius vector, scalar radius
$\mathbf{r}',r'$ = regularized velocity vector, radial component
R = Earth-moon distance
t = time
U = universal functions, Eq. (16)
$\mathbf{v},v$ = velocity vector, velocity
w = dummy variable
x = universal anomaly, Eq. (2)
y = dummy variable
α = $1/a$
$\delta(\ )$ = perturbation notation
$\boldsymbol{\varepsilon},\epsilon$ = eccentricity vector, eccentricity
θ = true anomaly
λ = normalized anomaly, Eq. (33)
μ = ν^2
ν = gravitational constant
σ = dummy variable
Φ,Ψ = state transition matrices
φ,ψ = state transition matrix elements
ω = orientation angle, Fig. 1

Superscripts and Subscripts

e = Earth
(F) = forced, or particular solution
(H) = homogeneous solution
m = moon
N = nonrotating coordinate system, Eq. (69)
o = initial condition
R = rotating coordinate system, Eq. (69)
T = matrix transpose
$\mathbf{x},\mathbf{y}$ = vector components along $\mathbf{i},\mathbf{j}$

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Introduction

THE advent of extensive manned exploration of the lunar surface has generated interest in the use of the cislunar-libration point L_1 as the possible location of a staging-space station. The relatively fixed location of the libration point L_1 in the rotating Earth-moon space at approximately 15% of the Earth-moon distance from the moon presents the distinct advantage of no time constraint on passage between L_1 and the moon or communication with the visible portion of the lunar surface. The feasibility of and the station-keeping requirements for such a space station, or libration point satellite, have been recently investigated by Farquhar[1] and others. This paper is concerned with the systematic development of a general regularized orbit perturbation theory with application to the analysis of near-parabolic trajectories between the moon and L_1.

The particular family of lunar trajectories investigated are Earth-perturbed, near-parabolic lunar orbits with perilune at the lunar surface. Although the theory may be extended to the elliptic or hyperbolic class of orbits, the choice of such would necessarily involve some considerations of available flight time vs available fuel and are beyond the scope of this analysis. Also, it might be noted that minimum energy-transfer trajectories between the moon and L_1 are necessarily near-parabolic. The assumption of perilune at the lunar surface is justified by the patching of the L_1-moon trajectory to a low-lunar circular orbit, consequently resulting in a more efficient over-all coverage of the lunar surface, as opposed to a direct ascent/descent modus operandi. The analysis considers trajectories passing both in front of and behind the moon and both to and from L_1.

The analysis of Earth-perturbed lunar orbits may be accomplished by numerical techniques usually involving the numerical integration of either Cowell- or Encke-type equations of motion, where numerical iterative techniques are generally used to resolve the two-point boundary value nature of the general transfer problem. Analytical representations in the time domain are generally complicated by a number of factors: 1) the analytical integrability of the perturbed two-body equations of motion, or the equivalent Encke perturbation equations; 2) the validity of an analytical theory for only a specific region of the elliptic/hyperbolic class of orbits, or difficulties with the transition from the elliptic to hyperbolic class; and 3) the occasional proximity of a solution, such as a series solution, to the singularity of the two-body equations of motion at $r_m = 0$.

The basis for the analytical treatment presented herein involves a local regularization, i.e., regularization with respect to the primary attracting point mass of the general unperturbed and perturbed two-body problems, first presented by Burdet[2,3] and arrived at subsequently and independently by the author, and the development of corresponding nonlinear and linearized perturbation equations. The resulting differential equations of motion are all nonsingular, and are universally applicable for any class of orbits, and the resulting linearized perturbation equations are analytically integrable for a variety of perturbing forces.

Regularization

The two-body vector orbit equation of motion for inverse square attraction between two bodies and with time as the independent variable is the familiar

$$\ddot{\mathbf{r}} = -\nu^2\mathbf{r}/r^3 \tag{1}$$

which is not only nonlinear and coupled but also singular at $r = 0$. The object of a regularization procedure is, of course, to remove the singularity of the vector equation; as will be subsequently seen, other undesirable characteristics of (1) are also eliminated by the regularization.

The procedure to be employed is the well-known change in the independent variable from time to a new variable x, the defining relation being

$$dx/dt = \nu/r \tag{2}$$

where, by quadrature, we obtain

ellipse: $x = (a)^{1/2}(E - E_o)$

hyperbola: $x = (|a|)^{1/2}(F - F_o)$

parabola: $x = (p)^{1/2}[\tan(\theta/2) - \tan(\theta_o/2)]$

rectilinear parabola: $x = [(2r)^{1/2} - (2r_o)^{1/2}]$

Regularization of Eq. (1) is then accomplished by first taking twice derivatives of $\mathbf{r}$

$$d(\mathbf{r})/dt = [d(\mathbf{r})/dx][dx/dt] \tag{3}$$

$$d^2(\mathbf{r})/dt = \mathbf{r}''\dot{x}^2 + \mathbf{r}'\ddot{x} = -\nu^2\mathbf{r}/r^3 \tag{4}$$

where

$$\ddot{x} = d[\nu/(\mathbf{r}\cdot\mathbf{r})^{1/2}]/dt = -\nu\mathbf{r}\cdot\dot{\mathbf{r}}/r^3 \tag{5}$$

Substitution of Eqs. (2) and (5) into Eq. (4) and multiplication by (r^2/ν^2) results in

$$\mathbf{r}'' + (-\mathbf{r}\cdot\dot{\mathbf{r}}/\nu^2)\dot{\mathbf{r}} + \mathbf{r}/r = 0 \tag{6}$$

At this stage it becomes necessary to derive formally the eccentricity vector $\boldsymbol{\varepsilon}$. From

$$d(\mathbf{r}/r)/dt = [(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}]/r^3 \tag{7}$$

$$= [(\mathbf{r}\times\dot{\mathbf{r}})\times\mathbf{r}]/r^3 \tag{7a}$$

$$= \mathbf{h}\times\mathbf{r}/r^3 \tag{7b}$$

$$= \ddot{\mathbf{r}}\times\mathbf{h}/\nu^2 \tag{7c}$$

we obtain

$$\mathbf{r}/r = \dot{\mathbf{r}}\times\mathbf{h}/\nu^2 - \mathbf{c} \tag{8}$$

where $\mathbf{c}$ is the vector constant of integration. Comparison of the dot product of Eq. (8) and $\mathbf{r}$ with well-known conic relations identifies the vector $\mathbf{c}$ to be the eccentricity vector $\boldsymbol{\varepsilon}$, a vector directed along periapsis with magnitude ϵ. Therefore

$$\boldsymbol{\varepsilon} = \dot{\mathbf{r}}\times\mathbf{h}/\nu^2 - \mathbf{r}/r \tag{9}$$

$$= (\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}/\nu^2)\mathbf{r} - (\mathbf{r}\cdot\dot{\mathbf{r}}/\nu^2)\dot{\mathbf{r}} - \mathbf{r}/r \tag{9a}$$

and the second term in the RHS of Eq. (9a) is recognized as the second term in Eq. (6); substitution yields

$$\mathbf{r}'' + (2/r - \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}/\nu^2)\mathbf{r} = -\boldsymbol{\varepsilon} \tag{10}$$

The scalar factor of $\mathbf{r}$ in (10) is recognized as the modified energy α $(=1/a)$, thus establishing the regularized vector orbit equation

$$\mathbf{r}'' + \alpha\mathbf{r} = -\boldsymbol{\varepsilon} \tag{11}$$

The resulting equation is not only nonsingular (as expected), but also a nondimensional, linear, constant coefficient vector equation. Since α is a scalar constant, the components of $\mathbf{r}$ are thus represented by uncoupled second-order systems. Moreover, both constants of Eq. (11) are orbit constants, and the type of conic, i.e., ellipse, parabola, hyperbola), directly corresponds to the type of Eq. (11) according to the signed value of α, recalling that

ellipse: $a > 0, \quad \alpha > 0$

parabola: $a = \infty, \quad \alpha = 0$

hyperbola: $a < 0, \quad \alpha < 0$

Taking the dot product of Eq. (11) with $\mathbf{r}$, and noting that

$$d(r)/dx = r' = \mathbf{r}'\cdot\mathbf{r}/r \tag{12}$$

and

$$d(\mathbf{r}'\cdot\mathbf{r})/dx = \mathbf{r}''\cdot\mathbf{r} + \mathbf{r}'\cdot\mathbf{r}' \tag{13}$$

$$= r''r + r'^2 \tag{13a}$$

we obtain a similar equation for the scalar radius

$$r'' + \alpha r = 1 \tag{14}$$

obtained earlier by Pitkin.[4]

The general solution to Eqs. (11, 14, and 2) is given by

$$\begin{pmatrix}\mathbf{r}\\ \mathbf{r}'\end{pmatrix} = \begin{pmatrix}U_1(x) & U_2(x) & -U_3(x)\\ -\alpha U_2(x) & U_1(x) & -U_2(x)\end{pmatrix}\begin{pmatrix}\mathbf{r}_o\\ \mathbf{r}_o'\\ \boldsymbol{\varepsilon}\end{pmatrix} \tag{15}$$

$$r = r_oU_1(x) + r_o'U_2(x) + U_3(x) \tag{15a}$$

$$\nu t = r_oU_2(x) + r_o'U_3(x) + U_4(x) \tag{15b}$$

where

$$U_1(x) = \cos(\alpha)^{1/2}x$$

$$U_2(x) = (a)^{1/2}\sin(\alpha)^{1/2}x$$

$$U_3(x) = a(1 - \cos(\alpha)^{1/2}x)$$

$$U_4(x) = a(x - (a)^{1/2}\sin(\alpha)^{1/2}x)$$

and corresponds to a different form of the universal orbit formulas of numerous investigators. The usual practice in the literature, (e.g., Battin[5]), is to define other special universal variables corresponding to the matrix elements of Eq. (15), namely

$$U_1(x) = \cos(\alpha)^{1/2}x = 1 - \alpha x^2/2! + \alpha^2x^4/4!\ldots = 1 - \alpha x^2C(\alpha x^2) \tag{16}$$

$$U_2(x) = (a)^{1/2}\sin(\alpha)^{1/2}x = x - \alpha x^3/3! + \alpha^2x^5/5!\ldots = x - \alpha x^3S(\alpha x^2) \tag{16a}$$

$$U_3(x) = a(1 - \cos(\alpha)^{1/2}x) = x^2/2! - \alpha x^4/4! + \alpha^2x^6/6!\ldots = x^2C(\alpha x^2) \tag{16b}$$

$$U_4(x) = a(x - (a)^{1/2}\sin(\alpha)^{1/2}x) = x^3/3! - \alpha x^5/5! + \alpha^2x^7/7!\ldots = x^3S(\alpha x^2) \tag{16c}$$

It might be noted that the U_j satisfy

$$U_{j}{}'_{+1}(x) = U_j(x)$$

Substitution of these special functions, along with (9) and (15b) evaluated at the initial conditions, yields the universal formulation given in Refs. 4 and 5.

The same mathematical treatment may be applied to the perturbed two-body problem, defined by

$$\ddot{\mathbf{r}} = -\nu^2\mathbf{r}/r^3 + \mathbf{f}(\mathbf{r}) \tag{17}$$

yielding

$$d^2\mathbf{r}/dx^2 + \alpha(x)\mathbf{r} = -\boldsymbol{\varepsilon}(x) + r^2\mathbf{f}(\mathbf{r})/\nu^2 \tag{18}$$

$$d^2r/dx + \alpha(x)r = 1 + r[\mathbf{r}\cdot\mathbf{f}(\mathbf{r})]/\nu^2 \tag{18a}$$

$$dt/dx = r/\nu \tag{18b}$$

which are nonlinear variable coefficient equations since $\alpha(x)$ and $\boldsymbol{\varepsilon}(x)$ are now slowly-varying orbit parameters whose differential equations are given by

$$d\alpha/dx = -2(d\mathbf{r}/dx)\cdot\mathbf{f}(\mathbf{r})/\nu^2 \tag{19}$$

$$d\boldsymbol{\varepsilon}/dx = \{2[(d\mathbf{r}/dx)\cdot\mathbf{f}(\mathbf{r})]\mathbf{r} - [\mathbf{r}\cdot\mathbf{f}(\mathbf{r})]d\mathbf{r}/dx - [\mathbf{r}\cdot d\mathbf{r}/dx]\mathbf{f}(\mathbf{r})\}/\nu^2 \tag{19a}$$

Burdet[3] resorts to a Cowell-type numerical integration of Eqs. (17–19) for analysis of motion about the triangular libration point L_4, and, along with other investigators, demonstrates the numerical adaptability and stability of the regularized two-body equations. In the next section, a general linearized perturbation theory is developed which retains, to some degree, the analytic simplicity and tractability of the unperturbed two-body equations.

Linearized Perturbation Equations

Linearized Encke-type perturbation equations, which represent the difference in a perturbed system state vector and some reference unperturbed conic state vector at the same value of the independent variable x, may be obtained directly by equating the first variation of the unperturbed system [Eqs. (11) and (14)] to the forcing functions of the perturbed system [Eqs. (18–19)], the latter terms evaluated along the unperturbed conic. The former involves a linear operation, and the latter is an approximation which assumes the perturbation state vector to be small in comparison to the unperturbed state vector.

The resulting linearized perturbation equations are

$$\delta\mathbf{r}^{*\prime\prime} + \alpha\delta r^* + \mathbf{r}\delta\alpha + \delta\boldsymbol{\varepsilon} = r^2\mathbf{f}(\mathbf{r})/\nu^2 \tag{20}$$

$$\delta r^{*\prime\prime} + \alpha\delta r^* + r\delta\alpha = r[\mathbf{r}\cdot\mathbf{f}(\mathbf{r})]/\nu^2 \tag{20a}$$

where

$$\delta\alpha' = -2\mathbf{r}'\cdot\mathbf{f}(\mathbf{r})/\nu^2 \tag{21}$$

$$\delta\boldsymbol{\varepsilon}' = \{2[\mathbf{r}'\cdot\mathbf{f}(\mathbf{r})]\mathbf{r} - [\mathbf{r}\cdot\mathbf{f}(\mathbf{r})]\mathbf{r}' - [\mathbf{r}\cdot\mathbf{r}']\mathbf{f}(\mathbf{r})\}/\nu^2 \tag{21a}$$

The asterisk superscript denotes what are referred to as fixed x perturbations, which compare the perturbed and unperturbed systems at the same value of the independent variable x. The corresponding difference in time, now regarded as a dependent variable, is then given by

$$\delta t' = \delta r^*/\nu \tag{22}$$

Equation (22) demonstrates the fact that time is progressing faster or slower along the perturbed trajectory than along the unperturbed trajectory; conversely, of course, the independent variables are different at the same time.

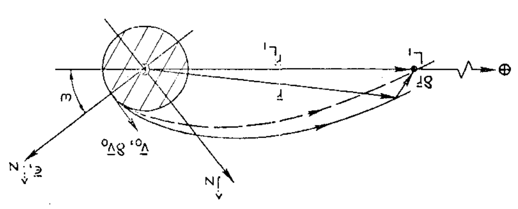

Fig. 1 Vector geometry for reference parabola relative to Earth-moon line.

Solution to Vector Perturbation Equation

The fixed x vector perturbation Eq. (20) may be written along with the auxiliary Eqs. (21) as

$$\frac{d}{dx}\begin{pmatrix}\delta\mathbf{r}^*\\ \delta\mathbf{r}^{*\prime}\\ \delta\boldsymbol{\varepsilon}\\ \delta\alpha\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ -\alpha & 0 & -1 & -\mathbf{r}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\delta\mathbf{r}^*\\ \delta\mathbf{r}^{*\prime}\\ \delta\boldsymbol{\varepsilon}\\ \delta\alpha\end{pmatrix} + \begin{pmatrix}0\\ r^2\mathbf{f}(\mathbf{r})/\nu^2\\ [2\mathbf{r}\mathbf{r}' - \mathbf{r}'\mathbf{r} - \mathbf{r}'\cdot\mathbf{r}I]\mathbf{f}(\mathbf{r})/\nu^2\\ -2\mathbf{r}'\cdot\mathbf{f}(\mathbf{r})/\nu^2\end{pmatrix} \tag{23}$$

where the notation $\mathbf{r}\mathbf{r}'$, e.g. denotes the dyad, or outer, product of the two vectors. Expressing (23) in the abbreviated form

$$d(\delta\mathbf{z}^*)/dx = F_v[\mathbf{r}(x)]\delta\mathbf{z}^* + \mathbf{g}_v\{\mathbf{r}(x),\mathbf{r}'(x),\mathbf{f}[\mathbf{r}(x)]\} \tag{23a}$$

the general solution to this linear, variable-coefficient system is given by

$$\delta\mathbf{z}^*(x) = \Phi[(x-0),\mathbf{r}(0),\mathbf{r}'(0)]\delta\mathbf{z}_o + \int_0^x \Phi[(x-\sigma),\mathbf{r}(\sigma),\mathbf{r}'(\sigma)]\mathbf{g}_v(\sigma)d\sigma \tag{24}$$

where

$$\Phi[w,\mathbf{r}(y),\mathbf{r}'(y)] = \begin{pmatrix}U_1(w) & U_2(w) & -U_3(w) & \boldsymbol{\phi}_{r\alpha}\\ -\alpha U_2(w) & U_1(w) & -U_2(w) & \boldsymbol{\phi}_{r'\alpha}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{25}$$

where $\omega = x - y$ and

$$\boldsymbol{\phi}_{r\alpha}[\omega,\mathbf{r}(y),\mathbf{r}'(y)] = -\mathbf{r}(y)(\omega/2)U_2(\omega) + \mathbf{r}'(y)(a/2)[\omega U_1(\omega) - U_2(\omega)] - a\boldsymbol{\varepsilon}[-U_3(\omega) + (\omega/2)U_2(\omega)] \tag{26}$$

$$\boldsymbol{\phi}_{r'\alpha}[\omega,\mathbf{r}(y),\mathbf{r}'(y)] = (\partial/\partial x)\boldsymbol{\phi}_{r\alpha}[\omega,\mathbf{r}(y),\mathbf{r}'(y)] \tag{26a}$$

The fixed x scalar perturbation Eq. (20a) may similarly be written as

$$\frac{d}{dx}\begin{pmatrix}\delta r^*\\ \delta r^{*\prime}\\ \delta\alpha\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -\alpha & 0 & -r\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\delta r^*\\ \delta r^{*\prime}\\ \delta\alpha\end{pmatrix} + \begin{pmatrix}0\\ r[\mathbf{r}.\mathbf{f}(\mathbf{r})]/\nu^2\\ -2\mathbf{r}'\cdot\mathbf{f}(\mathbf{r})/\nu^2\end{pmatrix} \tag{27}$$

or

$$d(\delta z^*)/dx = F_s[r(x)]\delta z^* + \mathbf{g}_s\{\mathbf{r}(x),\mathbf{r}'(x),\mathbf{f}[\mathbf{r}(x)]\} \tag{27a}$$

with the resulting solution

$$\delta z^*(x) = \Psi[(x-0),r(0),r'(0)]\delta z_o + \int_0^x \Psi[(x-\sigma),r(\sigma),r'(\sigma)]\mathbf{g}_s(\sigma)d\sigma \tag{28}$$

where

$$\Psi[\omega,r(y),r'(y)] = \begin{pmatrix}U_1(\omega) & U_2(\omega) & \psi_{r\alpha}\\ -\alpha U_2(\omega) & U_1(\omega) & \psi_{r'\alpha}\\ 0 & 0 & 1\end{pmatrix} \tag{29}$$

and

$$\psi_{r\alpha}[\omega,r(y),r'(y)] = -r(y)(\omega/2)U_2(\omega) + r'(y)(a/2)[\omega U_1(\omega) - U_2(\omega)] + a[-U_3(\omega) + (\omega/2)U_2(\omega)] \tag{29a}$$

$$\psi_{r'\alpha}[\omega,r(y),r'(y)] = (\partial/\partial x)\psi_{r\alpha}[\omega,r(y),r'(y)] \tag{29b}$$

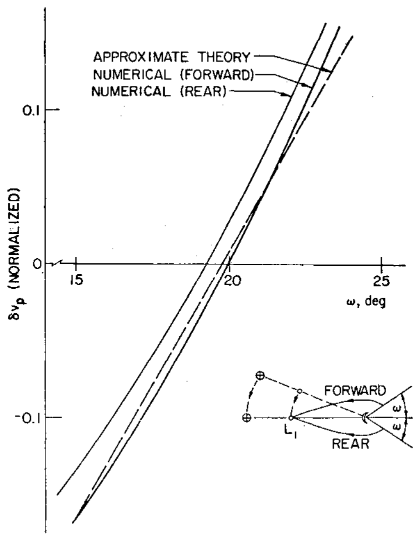

Fig. 2 Variation in parabolic velocity at perilune vs orientation angle ω.

A more complete discussion of the foregoing theory may be found in Ref. 6.

Reference Parabola

The state transition matrices (26) and (29) are simplified considerably by assuming the reference orbit to be a parabola, for which $\alpha = 0$. Thus

$$\Phi[w,\mathbf{r}(y),\mathbf{r}'(y)] = \begin{pmatrix} 1 & w & -w^2/2! & -\mathbf{r}(y)w^2/2! - \mathbf{r}'(y)w^3/3! + \boldsymbol{\varepsilon}w^4/4! \\ 0 & 1 & -w & -\mathbf{r}(y)w - \mathbf{r}'(y)w^2/2! + \boldsymbol{\varepsilon}w^3/3! \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (30)$$

and

$$\Psi[w,r(y),r'(y)] = \begin{pmatrix} 1 & \omega & -r(y)w^2/2! - r'(y)w^3/3! - w^4/4! \\ 0 & 1 & -r(y)w - r'(y)w^2/2! - w^3/3! \\ 0 & 0 & 1 \end{pmatrix} \quad (31)$$

L_1-Moon Trajectories

The application of the foregoing theory to the L_1-moon trajectory analysis is depicted in Fig. 1, where $\mathbf{r}$ represents an unperturbed parabola with perilune at the lunar surface and $\delta\mathbf{r}$ represents the combined perturbation position vector due to both the perturbing force and variations in the tangential perilune velocity δv_y, directed along $\mathbf{j}_N$.

The reference parabola is defined by the solutions to Eqs. (11, 14, and 2) for $\alpha = 0$,

$$\mathbf{r} = (p/2 - x^2/2,(p)^{1/2}x)^T \quad (32)$$

$$\mathbf{r}' = (-x,(p)^{1/2})^T \quad (32a)$$

$$r = p/2 + x^2/2 \quad (32b)$$

$$t = px/2 + x^3/6 \quad (32c)$$

Since $p = 2r_o$ for a parabola, solutions obtained in terms of x will be expressed through Eq. (32b) in the nondimensional form

$$\lambda = x/(p)^{1/2} = (r/r_o - 1)^{1/2} \quad (33)$$

The attraction of the Earth is given by

$$\mathbf{f}(\mathbf{r}) = \mu_e[(\mathbf{R} - \mathbf{r})/|\mathbf{R} - \mathbf{r}|^3 - \mathbf{R}/R^3] \quad (34)$$

Expanding (34) in ascending powers of $\mathbf{r}$ and assuming the Earth to be fixed along the final orientation of the Earth-L_1-moon line in the nonrotating lunar space, the attraction of the Earth $\mathbf{f}(\mathbf{r})$ is approximated to first order in $\mathbf{r}$ by

$$\mathbf{f}(\mathbf{r}) = \frac{\mu_e}{2R^3}\begin{pmatrix} 3\cos 2\omega + 1 & -3\sin 2\omega \\ -3\sin 2\omega & -3\cos 2\omega + 1 \end{pmatrix}\mathbf{r} \quad (35)$$

Since the analysis is considering only variations in the initial perilune velocity vector, (i.e., $\delta\mathbf{r}_o = 0$), expressed in the regularized domain as

$$\delta\mathbf{r}_o' = \delta\mathbf{v}_o/\dot{x}_o \quad (36)$$

the terms $\delta\alpha_o$ and $\delta\boldsymbol{\varepsilon}_o$ in Eqs. (24) and (28) may be expressed simply as

$$\delta\alpha_o = -2\mathbf{r}_o'\cdot\delta\mathbf{r}_o'/r_o^2 \quad (37)$$

$$\delta\boldsymbol{\varepsilon}_o = [2\mathbf{r}_o\mathbf{r}_o' - \mathbf{r}_o'\mathbf{r}_o - \mathbf{r}_o'\cdot\mathbf{r}_o I]\delta\mathbf{r}_o'/r_o^2 \quad (37a)$$

Substitution of Eqs. (30) and (37) into Eq. (24) results in a somewhat simpler form for the integration of $\delta\mathbf{r}^*$, recalling that $\mathbf{r}'' = -\boldsymbol{\varepsilon}$ for the parabola

$$\begin{aligned}\delta\mathbf{r}^* = \{[r_o^2 I]x + [\mathbf{r}_o'\mathbf{r}_o + \mathbf{r}_o'\cdot\mathbf{r}_o I]x^2/2! + [\mathbf{r}_o'\mathbf{r}_o']2x^3/3! + \\ [\mathbf{r}_o''\mathbf{r}_o']2x^4/4!\}\delta\mathbf{r}_o'/r_o^2 + \int_0^x \{[r^2(\sigma)I](x-\sigma) + \\ [\mathbf{r}'(\sigma)\mathbf{r}(\sigma) + \mathbf{r}'(\sigma)\cdot\mathbf{r}(\sigma)I](x-\sigma)^2/2! + \\ [\mathbf{r}'(\sigma)\mathbf{r}'(\sigma)]2(x-\sigma)^3/3! + [\mathbf{r}''(\sigma)\mathbf{r}'(\sigma)]2(x-\sigma)^4/4!\} \times \\ \mathbf{f}[\mathbf{r}(\sigma)]d\sigma/\mu_m \quad (38)\end{aligned}$$

Using (32) and

$$\delta\mathbf{r}_o' = [0,\delta r_y']^T \quad (39)$$

where

$$\delta r_y' = \delta v_y/\dot{x}_0 \quad (40)$$

the homogeneous part of (38) reduces to

$$\delta\mathbf{r}_{(H)}^* = [\phi_{xv}(\lambda),\phi_{yv}(\lambda)]^T\delta r_y' \quad (41)$$

where

$$\phi_{xv}(\lambda) = -(p)^{1/2}\lambda^4/3 \quad (42)$$

$$\phi_{yv}(\lambda) = (p)^{1/2}\lambda(4\lambda^2 + 3)/3 \quad (42a)$$

Equivalently, changes in the final fixed x regularized velocity vector are related to $\delta r_y'$ by

$$\delta\mathbf{r}_{(H)}^{*\prime} = [\phi_{xv}'(\lambda),\phi_{yv}'(\lambda)]^T\delta r_y' \quad (43)$$

where

$$\phi_{xv}'(\lambda) = -4\lambda^3/3 \quad (44)$$

$$\phi_{yv}'(\lambda) = 4\lambda^2 + 1 \quad (44a)$$

Using (32), the integrand of (38) may be expanded in ascending powers of $(p)^{1/2}$ up to and truncating at $\mathcal{O}(p^3)$, and the integral may then be expressed in descending powers of the nondimensional parameter λ. Retaining only the highest power of λ [since $x \gg (p)^{1/2}$ in vicinity of L_1], the particular solution to Eq. (38) is

$$\frac{\delta\mathbf{r}_{(F)}^*}{r_0} = \left(\frac{\mu_e}{\mu_m}\right)\left(\frac{r_0}{R}\right)^3\begin{pmatrix} -\lambda^8[3\cos 2\omega + 1]/28 \\ \lambda^8\sin 2\omega/14 \end{pmatrix} \quad (45)$$

and

$$\frac{\delta\mathbf{r}_{(F)}^{*\prime}}{(p)^{1/2}} = \left(\frac{\mu_e}{\mu_m}\right)\left(\frac{r_o}{R}\right)^3\begin{pmatrix} -\lambda^7[3\cos 2\omega + 1]/7 \\ \lambda^7[2\sin 2\omega]/7 \end{pmatrix} \quad (46)$$

The homogeneous solution to the scalar perturbation Eq. (28) is simplified by noting that for

$$\delta\mathbf{r}_o = \delta r_o = 0 \quad (47)$$

$$\mathbf{r}_o\cdot\mathbf{r}_o' = 0 \tag{47a}$$

$$\mathbf{r}_o\cdot\delta\mathbf{r}_o' = 0 \tag{47b}$$

we obtain

$$r_o' = \mathbf{r}_o\cdot\mathbf{r}_o'/r_o = 0 \tag{48}$$

$$\delta r_o' = (\mathbf{r}_o'\cdot\delta\mathbf{r}_o + \mathbf{r}_o\cdot\delta\mathbf{r}_o')/r_s - [(\mathbf{r}_o\cdot\delta\mathbf{r}_o)(\mathbf{r}_o\cdot\mathbf{r}_o')]/r_o^3 \tag{48a}$$

Thus, using Eq. (31), we obtain

$$\delta\mathbf{r}_{(H)}^* = -p^2(\lambda^2/4 + \lambda^4/4!)\delta\alpha_o \tag{49}$$

where

$$\delta\alpha_o = -2\mathbf{r}_o'\cdot\delta\mathbf{r}_o'/r_o^2 = -2(p)^{1/2}\delta r_y'/r_o^2 \tag{50}$$

and

$$\delta r_{(H)}^{*\prime} = -(p)^{3/2}(\lambda^2/2 + \lambda^3/3!)\delta\alpha_o \tag{51}$$

The particular solution to Eq. (28), retaining only the term of the highest power of λ, is given by

$$\delta r_{(F)}^*/r_o = (\mu_e/\mu_m)(r_o/R)^3(3\cos 2\omega + 1)\lambda^8/28 \tag{52}$$

The difference in time between the perturbed and unperturbed systems is then obtained from Eq. (22)

$$\delta t' = (\delta r_{(H)}^* + \delta r_{(F)}^*)/\nu_m \tag{53}$$

$$\delta t = \{p(2\lambda^3/3 + \lambda^5/15)\delta r_y' + (p)^{3/2}(\mu_e/\mu_m)(r_o/R)^3(3\cos 2\omega + 1)\lambda^9/512\}/\nu_m \tag{53a}$$

Referring now to Fig. 1,

$$\mathbf{r}_{L_1}(x) = \mathbf{r}(x) + \delta\mathbf{r}_{(H)}^*(x) + \delta\mathbf{r}_{(F)}^*(x) \tag{54}$$

Using the expansion

$$x = x_{L_1} + \delta x \tag{55}$$

and

$$\mathbf{r}(x) = \mathbf{r}(x_{L_1}) + \mathbf{r}'(x_{L_1})\delta x \tag{56}$$

where

$$x_{L_1} = (2r_{L_1} - p)^{1/2} = (p)^{1/2}\lambda_{L_1} \tag{57}$$

and the approximations

$$\delta\mathbf{r}_{(H)}^*(\lambda) \simeq \delta\mathbf{r}_{(H)}^*(\lambda_{L_1}) \tag{58}$$

$$\delta\mathbf{r}_{(F)}^*(\lambda) \simeq \delta\mathbf{r}_{(F)}^*(\lambda_{L_1}) \tag{58a}$$

Equation (54) becomes

$$\begin{pmatrix} -r_{L_1}\cos\omega \\ r_{L_1}\sin\omega \end{pmatrix} = \begin{pmatrix} r_0 - x_{L_1}^2/2 \\ (p)^{1/2}x_{L_1} \end{pmatrix} + \begin{pmatrix} -x_{L_1} \\ (p)^{1/2} \end{pmatrix}\delta x + \delta\mathbf{r}_{(F)}^* + \begin{pmatrix} \phi_{xv}(x_{L_1}) \\ \phi_{yv}(x_{L_1}) \end{pmatrix}\delta r_y' \tag{59}$$

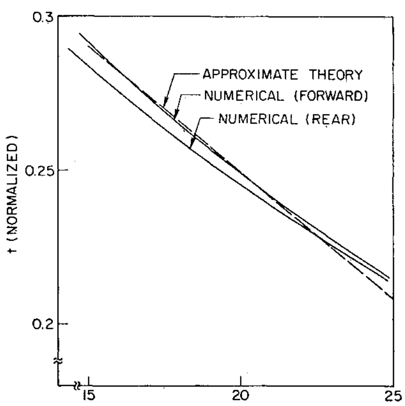

Fig. 3 Transit time between perilune and L_1 vs orientation angle ω.

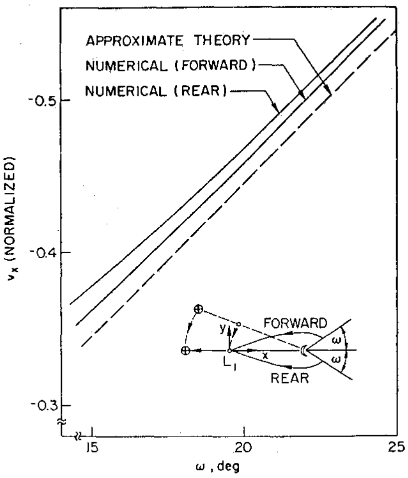

Fig. 4 Axial-velocity component V_x at L_1 vs orientation angle ω.

and the terms $\delta r_y'$ and δx are obtainable from

$$\begin{pmatrix} \delta r_y' \\ \delta x \end{pmatrix} = \frac{1}{(p)^{1/2}\phi_{xv}(x_{L_1}) + x_{L_1}\phi_{yv}(x_{L_1})} \times \begin{pmatrix} (p)^{1/2} & x_{L1} \\ -\phi_{yv}(x_{L_1}) & \phi_{xv}(x_{L_1}) \end{pmatrix}\left\{\begin{pmatrix} -r_{L_1}\cos\omega - r_0 + x_{L_1}^2/2 \\ r_{L_1}\sin\omega - (p)^{1/2}x_{L_1} \end{pmatrix} - \delta\mathbf{r}_{(F)}^*\right\} \tag{60}$$

From Eqs. (42) and (45) and noting that

$$(p)^{1/2}\phi_{xv}(x_{L_1}) + x_{L1}\phi_{yv}(x_{L_1}) = p\lambda_{L_1}^2(\lambda_{L_1}^2 + 1) = 2r_{L_1}\lambda_{L_1}^2 \tag{61}$$

we obtain an approximate relation for the variation in perilune velocity as a function of the orientation angle ω

$$\delta r_y'/(p)^{1/2} = \delta v_y/v_y = \{\lambda_{L_1}\sin\omega - \cos\omega + 1 + (\mu_e/\mu_m)(r_o/R)^3(r_o/r_{L_1})[\lambda_{L_1}^8(3\cos 2\omega + 1)/28 - \lambda_{L_1}^9\sin 2\omega/14]\}/2\lambda_{L_1}^2 \tag{62}$$

The term δx may be expressed as

$$\delta x/(p)^{1/2} = \delta\lambda = \{(4\lambda_{L_1}^2 + 3)[r_{L_1}(\cos\omega + 1) + p + \delta\mathbf{r}_{(F)x}^*] - \lambda_{L_1}^3[r_{L_1}\sin\omega - p\lambda_{L_1} - \delta\mathbf{r}_{(F)y}^*]\}/6\lambda_{L_1}r_{L_1} \tag{63}$$

The resultant velocity vector of arrival at L_1 in the nonrotating space is then

$$\mathbf{v}_{L_1} = \mathbf{r}_{L_1}'\dot{x}_{L_1} = \mathbf{r}_{L_1}'\nu_m/r_{L_1} \tag{64}$$

where

$$\mathbf{r}_{L_1}'(x_{L_1}) = \mathbf{r}'(x_{L_1}) + \mathbf{r}''(x_{L_1})\delta x + \delta\mathbf{r}_{(H)}^*(x_{L_1}) + \delta\mathbf{r}_{(F)}^*(x_{L_1}) \tag{65}$$

The time of arrival at L_1 is obtained from

$$t_{L_1} = t(x_{L_1}) + \delta t(x_{L_1}) + \delta x(x_{L_1})/\dot{x}_{L_1} \tag{66}$$

where δt is obtained from Eq. (53a).

Results

All parameters have been normalized to the dimensions of the Earth-moon system, namely,

$$\begin{aligned} &\text{Length:} \quad R = 1 \\ &\text{Mass:} \quad \mu_e + \mu_m = 1 \\ &\text{Time} \quad n^2 = 1 = (\mu_e + \mu_m)/R^3 \end{aligned} \tag{67}$$

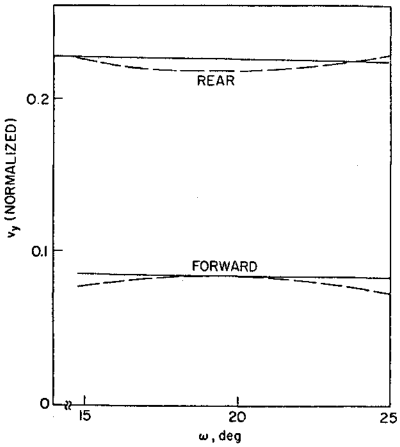

Fig. 5 Lateral-velocity component V_y at L_1 vs orientation angle ω.

Normalized velocities are then related to unnormalized velocities through

$$v = [(\mu_e + \mu_m)/R]^{1/2} v_{\text{norm}} \tag{68}$$

and a unit of time is equal to $P_m/2\pi$, or approximately $4\frac{1}{3}$ days. The value for μ_m was taken to be 0.0121507, corresponding to $\mu_e/\mu_m \simeq 81.3$, with $r_{L_1} = 0.151$ and $r_o = 0.00427$ (1740 km) (Ref. 1). All calculations were made for the idealized planar Earth-moon system, i.e., both bodies in circular orbits about the barycenter.

The exact nonlinear perturbation equations were numerically integrated for lunar orbits from the moon to L_1, passing both in front of and behind the moon. Because of the reflection property of Earth-moon trajectories in the earth-moon line of the rotating frame (exact), the velocity vector of departure (arrival) in the rotating frame for passage behind the moon is the reflection of the velocity vector of arrival (departure) for passage in front of the moon. With regard to the analytical theory, the simplifying assumption of the fixed Earth renders the departure and arrival-velocity vectors identical, for passage either in front of or behind the moon.

Figure 2 presents the variation in δv_y as a function of ω, δv_y being related to parabolic velocity v_y at perilune through Eq. (62). The variation in total transit time, Eq. (66), as a function of ω is shown in Fig. 3. It might be noted that the lunar longitude at the time of perilune passage must necessarily be obtained from

$$\text{Longitude} = \pi - \omega - t_{L_1}(\omega)$$

The velocity vector of arrival/departure in the rotating Earth-moon space is

$$\underset{R}{\mathbf{v}_{L1}} = \pm \left[C_{R/N} \underset{N}{\mathbf{v}_{L1}} + \mathbf{n} \times \underset{R}{\mathbf{r}_{L1}} \right] \tag{69}$$

where $+$ = arrival, $-$ = departure, $\mathbf{n} = \mathbf{k}_R = \mathbf{k}_N$, $\underset{R}{\mathbf{r}_{L_1}} = r_{L_1}\mathbf{i}_R$, and

$$C_{R/N} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \pm\sin\omega & \pm\cos\omega \end{pmatrix} \tag{70}$$

where $+$ = passage in front of moon, and $-$ = passage behind moon. The arrival-velocity component v_x along $\mathbf{i}_R$ is given in Fig. 4, and the lateral-arrival velocity component v_y is shown in Fig. 5 for passage in front of and behind the moon. The differences in the numerical integrations may be viewed, in the nonrotating frame, as the difference in geometry of the Earth attraction relative to the lunar orbit, while in the rotating frame simply reflect the fact that passage in front of the moon takes advantage of the moon's motion about L_1. The degree of agreement between the approximate theory and either of the numerically integrated data would seem to depend on the validity of the approximate assumed geometry relative to either of the exact geometries of the Earth attraction.

Conclusions

A linearized-analytical-perturbation theory, based on regularization of the perturbed two-body problem, has been presented and applied to the analysis of a particular class of trajectories between the moon and the cislunar-libration point L_1. The approximate relations presented appear to yield close agreement to corresponding numerical data. Because of the relative simplicity of analytical manipulations in the regularized domain, numerous refinements to the approximations used herein might be used to refine further the analytical representation of the class of trajectories investigated. In addition, other classes of reference orbits might be investigated, including the general rectilinear orbits. One interesting feature of analysis in the regularized domain is that both time-varying and position-dependent effects may be easily investigated since both time and position are explicit functions of the independent variable.

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